Presentation
Gradient estimate for double-phase problems with logarithmic growth

AARG Seminar - Ton Duc Thang University

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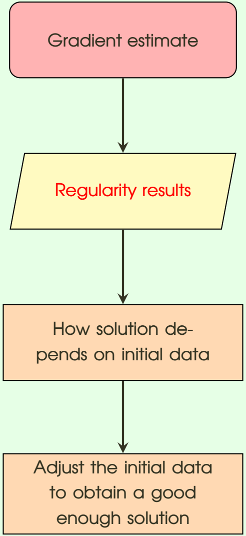
Contents

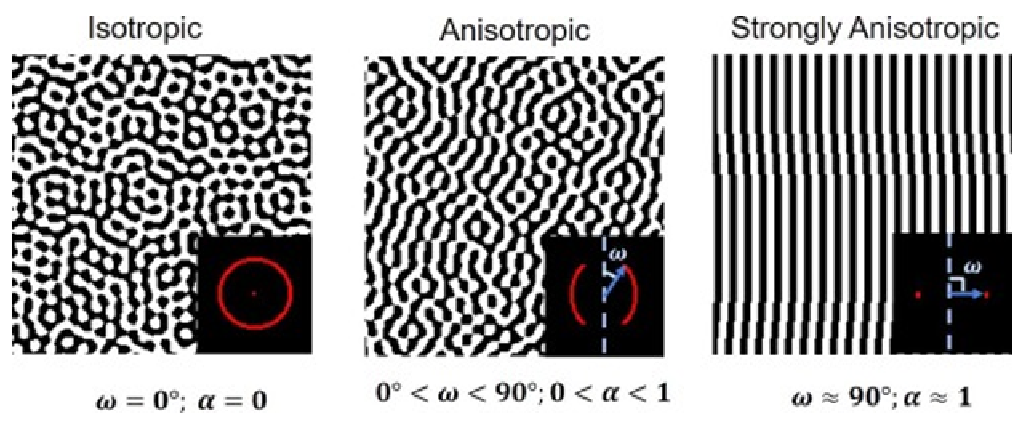
Regularity results

Why we need to study regularity results?



Double-phase problems

Application to anisotropic materials



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Double-phase problems

Application to non-Newtonian fluids

Y. Lu, Y. Liu, X. Huang, C. Vetro, *A new kind of double phase elliptic inclusions with logarithmic perturbation terms II: applications*, Commun. Nonlinear Sci. Numer. Simul., **131** (2024).

Double-phase problems

Application to image processing

A. Charkaoui, *Nonlinear parabolic double phase variable exponent systems with applications in image noise removal*, Appl. Math. Comput., **467** (2024).

Double phase problems with logarithmic growth

Previous work

$$-\operatorname{div}\left(\mathcal{A}(x,\nabla u)\right) = -\operatorname{div}\left(\mathcal{B}(x,\mathbf{F})\right), \quad x \in \Omega \tag{P}$$

In (Byun '17), under suitable assumptions on $\mathcal{A}$, $\mathcal{B}$ and Ω, Byun et al. have proved the so-called Calderón-Zygmund estimate or gradient estimate in Lebesgue space $L^\gamma(\Omega)$:

For all $\gamma > 1$, we have:

$$\mathbb{F} := \mathbb{H}_{p,\log}(\cdot,\mathbf{F}) \in L^\gamma(\Omega) \Longrightarrow \mathbb{U} := \mathbb{H}_{p,\log}(\cdot,\nabla u) \in L^\gamma(\Omega). \tag{CZ}$$

S. S. Byun, J. Oh, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations, **263**(2) (2017), 1643-1693.

Double-phase problems with logarithmic growth

Problem statement

$$\begin{cases} -\mathrm{div}\left(\mathcal{A}_{p,\log}(x,\nabla u)\right) = -\mathrm{div}\left(\mathcal{B}_{p,\log}(x,\mathbf{F})\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{DPL}$$

under assumptions

$$\begin{cases} |\mathcal{A}_{p,\log}(x,\xi)| + |\partial_\xi \mathcal{A}_{p,\log}(x,\xi)||\xi| \le K_1 \mathbb{H}_{p-1,\log}(x,\xi); \\ \langle \partial_\xi \mathcal{A}_{p,\log}(x,\xi) z, z\rangle \ge K_2 \mathbb{H}_{p-2,\log}(x,\xi)|z|^2, \end{cases} \tag{A1}$$

$$|\mathcal{B}_{p,\log}(x,\xi)| \le K_3 \mathbb{H}_{p-1,\log}(x,\xi), \tag{A2}$$

with

$$\mathbb{H}_{q,\log}(x,\xi) := |\xi|^q + a(x)|\xi|^q \log(e+|\xi|), \quad (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{H}$$

Double-phase problems with logarithmic growth

Problem model

The above problem is a generic one whose model is given by

$$-\mathrm{div}\left(\beta(x)\left[|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{p-2}\log(e+|\nabla u|)\nabla u\right]\right)$$
$$= -\mathrm{div}\left(|\mathbf{F}|^{p-2}\mathbf{F} + a(x)|\mathbf{F}|^{p-2}\log(e+|\mathbf{F}|)\mathbf{F}\right) \quad \text{in } \Omega,$$

which is relevant to the corresponding Euler-Lagrange equation of functional

$$v \mapsto \mathcal{H}(v) - \int_\Omega \left\langle |F|^{p-2}F + a(x)|F|^{p-2}\log(e+|F|)F, \nabla v \right\rangle dx,$$

where the energy functional $\mathcal{H}$ is given by

$$\mathcal{H}(v) := \int_\Omega \beta(x)\left[|\nabla v|^p + a(x)|\nabla v|^p \log(e+|\nabla v|)\right] dx.$$

Double-phase problems with logarithmic growth

Weak solution

Definition (Weak solution)

A function $u \in W_0^{1,\mathbb{H}_{p,\log}}(\Omega)$ is said to be weak solution to (DPL) if the following variational formula holds for every test function $\varphi \in W_0^{1,\mathbb{H}_{p,\log}}(\Omega)$:

$$\int_\Omega \langle \mathcal{A}_{p,\log}(x, \nabla u), \nabla \varphi \rangle dx = \int_\Omega \langle \mathcal{B}_{p,\log}(x, \mathbf{F}), \nabla \varphi \rangle dx. \tag{VF}$$

Double-phase problems with logarithmic growth

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It is well-known that (DPL) admits an unique weak solution $u \in W_0^{1,\mathbb{H}_{p,\log}}(\Omega)$ which also satisfies the following global estimate, proved in (Byun '17, Theorem 3.6):

$$\int_\Omega \mathbb{H}_{p,\log}(x, \nabla u) dx \leq C \int_\Omega \mathbb{H}_{p,\log}(x, \mathbf{F}) dx. \tag{GE}$$

S. S. Byun, J. Oh, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations, **263**(2) (2017), 1643-1693.

Double-phase problems with logarithmic growth

Our regularity results

Our main result is a gradient estimate for a borderline case of double-phase problems with logarithmic growth in generalized weighted Lorentz spaces $\Lambda^{q,\mathfrak{s}}_{\omega,\mu}(\Omega)$, extending the result in (Byun '17) to a gradient estimate via fractional maximal operators $\mathbb{M}_\beta$ as follows:

$$\mathbb{M}_\beta \mathbb{F} \in \Lambda^{q,\mathfrak{s}}_{\omega,\mu}(\Omega) \Longrightarrow \mathbb{M}_\beta \mathbb{U} \in \Lambda^{q,\mathfrak{s}}_{\omega,\mu}(\Omega). \tag{1}$$

More precisely, we establish the gradient estimate based on the good-λ type inequality introduced in the sequel:

$$\|\mathbb{M}_\beta \mathbb{U}\|_{\Lambda^{q,\mathfrak{s}}_{\omega,\mu}(\Omega)} \leq C\|\mathbb{M}_\beta \mathbb{F}\|_{\Lambda^{q,\mathfrak{s}}_{\omega,\mu}(\Omega)}, \tag{2}$$

S. S. Byun, J. Oh, *Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains*, J. Differential Equations, **263**(2) (2017), 1643-1693.

Contents

Fractional maximal operator

Definition and boundedness

Definition (Fractional maximal operator)

The maximal operator $\mathbb{M}_\beta$ with $\beta \in [0, n]$ is a measurable map defined on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ reads as follows:

$$\mathbb{M}_\beta \mathrm{f}(x) = \sup_{B \ni x} \, [r(B)]^\beta \fint_B |\mathrm{f}(\zeta)| d\zeta, \quad x \in \mathbb{R}^n, \ \mathrm{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{3}$$

It is readily seen that $\mathbb{M}_\beta$ coincides with the classical Hardy-Littlewood maximal operator $\mathbb{M}$ when $\beta = 0$.

Lemma (Boundedness of fractional maximal operator)

Let $\mathfrak{q} \geq 1$ and $\beta \in \left[0, \dfrac{n}{\mathfrak{q}}\right)$. There exists $C = C(n, \mathfrak{q}, \beta) > 0$ such that

$$\left|\{x \in \mathbb{R}^n : \ \mathbb{M}_\beta \mathrm{f}(x) > \lambda\}\right| \leq C \left(\frac{1}{\lambda^{\mathfrak{q}}} \int_{\mathbb{R}^n} |\mathrm{f}(\zeta)|^{\mathfrak{q}} d\zeta\right)^{\frac{n}{n - \mathfrak{q}\beta}}, \quad \text{for all } \lambda > 0, \ \mathrm{f} \in L^{\mathfrak{q}}(\mathbb{R}^n).$$

Fractional maximal operator

Why fractional maximal operator comes into play?

Fractional maximal function has a relation to Riesz potential (fractional integral operator) due to the following point-wise inequality:

$$\mathbb{M}_a f(x) \leq C \cdot \mathbb{I}_a f(x), \quad \text{for every } x \in \mathbb{R}^n.$$

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Tan-Phuc Nguyen's Presentation

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$$\mathbb{M}_\alpha f(x) \le C \cdot \mathbb{I}_\alpha f(x), \quad \text{for every } x \in \mathbb{R}^n.$$

Moreover, as shown in (Muckenhoupt '74, Theorem 2.1), the converse inequality holds in its integral form as below:

$$\int_{\mathbb{R}^n} (\mathbb{I}_\alpha \mathbf{f})^q \omega dx \le C \int_{\mathbb{R}^n} (\mathbb{M}_\alpha \mathbf{f})^q \omega dx, \quad \text{for } q > 0 \text{ and } \omega \in \mathbf{A}_\infty.$$

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Therefore, gradient estimates for solutions to elliptic problems via fractional maximal functions not only provide information of size and oscillations of solutions and their derivatives, but also allow to bound fractional derivatives of u: $\partial^\alpha u$, for $0 \le \alpha < 2$.

Muckenhoupt, R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**(1974), 261-274.

Muckenhoupt class

Defintion

By a weight, we mean a locally integrable function μ on $\mathbb{R}^n$, i.e. $\mu \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^+)$. For each weight μ, it gives a measure on the measurable subsets of $\mathbb{R}^n$ through integration. For every measurable set $\mathfrak{D} \subset \mathbb{R}^n$, we denote $\mu(\mathfrak{D}) := \int_{\mathfrak{D}} \mu(\zeta) d\zeta$. In a special case when $\mu \equiv 1$, it is well-known that $\mu(\mathfrak{D}) \equiv |\mathfrak{D}|$.

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Definition (Muckenhoupt class)

We say that a weighted function μ belonging to the Muckenhoupt class $\mathbf{A}_\infty$ if there are constants c_1, c_2 and $\upsilon_1, \upsilon_2 > 0$ such that for every measurable subset $\mathfrak{D}$ of a ball $\mathbb{B}$:

$$c_1 \left[\frac{|\mathfrak{D}|}{|\mathbb{B}|}\right]^{\upsilon_1} \leq \frac{\mu(\mathfrak{D})}{\mu(\mathbb{B})} \leq c_2 \left[\frac{|\mathfrak{D}|}{|\mathbb{B}|}\right]^{\upsilon_2}, \tag{4}$$

When no ambiguity will arise, we denote $[\mu]_{\mathbf{A}_\infty} = (c_1, c_2, \upsilon_1, \upsilon_2)$.

Generalized weighted Lorentz spaces

Definition

Definition (Generalized weighted Lorentz spaces)

Consider two weights $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^+)$, $\omega \in \mathbf{A}_\infty$ and a function Γ defined as follows $\Gamma(\tau) = \displaystyle\int_0^\tau \mu(s)ds$, $\tau \in [0,\infty)$. Given $\mathfrak{q} \in (0,\infty)$ and $0 < \mathfrak{s} \leq \infty$, the generalized weighted Lorentz space $\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)$ is defined by:

$$\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega) := \left\{f : \Omega \to \mathbb{R} \text{ is Lebesgue measurable and } \|f\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)} < \infty\right\}, \tag{5}$$

where the quasi-norm $\|\cdot\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)}$ is given by:

$$\|f\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)} := \left\{\mathfrak{q}\int_0^\infty \left[\lambda^{\mathfrak{q}}\Gamma\left(\omega(\{x \in \Omega : |f(x)| > \lambda\})\right)\right]^{\frac{\mathfrak{s}}{\mathfrak{q}}} \frac{d\lambda}{\lambda}\right\}^{\frac{1}{\mathfrak{s}}}, \quad \text{if} \quad s < \infty$$

$$\|f\|_{\Lambda^{\mathfrak{q},\infty}_{\omega,\mu}(\Omega)} := \sup_{\lambda>0}\left[\lambda^{\mathfrak{q}}\Gamma\left(\omega(\{x \in \Omega : |f(x)| > \lambda\})\right)\right]^{\frac{1}{\mathfrak{q}}}, \quad \text{if} \quad s = \infty$$

Contents

Framework

Main steps

$\mathbb{H}_{p,\log}$ estimates

↓ Assumptions on $\mathcal{A}_{p,\log}$, $\mathcal{B}_{p,\log}$ and suitable test function

Comparision estimate

↓ Reverse Hölder inequality, Vitali's covering lemma, boundedness of fractional maximal operators

Good-λ type inequality

↓ Definition of quasi-norm in generalized weighted Lorentz spaces

Generalized weighted Lorentz estimate

$\mathbb{H}_{p,\log}$ estimates

Define auxiliary functions

$$\mathbb{V}(x,\xi_1,\xi_2) := |V_p(\xi_1) - V_p(\xi_2)|^2 + a(x)|V_{p\log}(\xi_1) - V_{p\log}(\xi_2)|^2, \ (x,\xi_1,\xi_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$$

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}}\xi, \ V_{p\log}(\xi) := \left(|\xi|^{p-2}\log(e+|\xi|) + \frac{|\xi|^{p-1}}{p(e+|\xi|)}\right)^{\frac{1}{2}}\xi, \quad \xi \in \mathbb{R}^n.$$

Lemma ($\mathbb{H}_{p,\log}$ estimates)

For every $p > 1$ and $\epsilon \in (0,1)$, the following estimates

$$|\xi_1|\mathbb{H}_{p-1,\log}(x,\xi_2) \le \epsilon\mathbb{H}_{p,\log}(x,\xi_1) + \epsilon^{-\frac{1}{p-1}}\mathbb{H}_{p,\log}(x,\xi_2), \tag{6}$$

and

$$\mathbb{H}_{p,\log}(x,\xi_1-\xi_2) \le \epsilon\mathbb{H}_{p,\log}(x,\xi_1) + C\epsilon^{-1}\mathbb{V}(x,\xi_1,\xi_2), \tag{7}$$

hold true for all $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^n$, where $C = C(p) > 0$.

Comparison estimate and reverse Hölder inequality

Lemma (Comparison estimate and reverse Hölder inequality)

For every $x_0 \in \overline{\Omega}$, $r > 0$, there exists $\delta_0 > 0$ sufficiently small such that if $\mathcal{A}_{p,\log}$ is (δ_0, R_0)-vanishing, Ω is (δ_0, R_0)-Reifenberg flat for some $R_0 > 0$, then one can find $v \in W^{1,\mathbb{H}_{p,\log}}(\Omega(x_0, 4r))$ satisfying

$$\left[\fint_{\Omega(x_0,4r)} \left[\mathbb{H}_{p,\log}(x, \nabla v)\right]^{\gamma} dx\right]^{\frac{1}{\gamma}} \leq C \fint_{\Omega(x_0,8r)} \mathbb{H}_{p,\log}(x, \nabla v)dx, \tag{8}$$

for each $\gamma > 1$, where $C = C(\mathtt{data}) > 0$. Moreover, for every $\varepsilon \in (0,1)$, there exists constants $\kappa := \dfrac{p+1}{p-1} > 0$ and $C = C(\mathtt{data}) > 0$ satisfying

$$\fint_{\Omega(x_0,8r)} \mathbb{H}_{p,\log}(x, \nabla u - \nabla v)dx \leq \varepsilon \fint_{\Omega(x_0,8r)} \mathbb{H}_{p,\log}(x, \nabla u)dx + C\varepsilon^{-\kappa} \fint_{\Omega(x_0,8r)} \mathbb{H}_{p,\log}(x, \mathbf{F})dx. \tag{9}$$

Good-$\mathcal{λ}$ type inequality

Lemma (Vitali's covering lemma)

Assume that Ω is a domain satisfied the (δ, R_0)-Reifenberg condition for some constants $R_0, \delta > 0$ and $\omega \in \mathbf{A}_\infty$. Suppose $0 < \varepsilon < 1$, $0 < R \leq R_0$ and consider two measurable sets $\mathcal{V}, \mathcal{W}$ satisfying $\mathcal{V} \subset \mathcal{W} \subset \Omega$ and the following two conditions:

(i) $\omega(\mathcal{V}) \leq \varepsilon\omega\left(B(0,R)\right)$;

(ii) For all $x \in \Omega$ and $r \in (0, R]$, we have $B(x,r) \cap \Omega \subset \mathcal{W}$ if $\omega\left(V \cap B(x,r)\right) > \varepsilon\omega\left(B(x,r)\right)$.

Then $\omega(\mathcal{V}) \leq C\varepsilon\omega(\mathcal{W})$ for some constant $C = C(n) > 0$.

Theorem (Good-$\mathcal{λ}$ type inequality)

Given $\beta \in [0,n)$, $\omega \in \mathbf{A}_\infty$. Then, for every $a \in (0,1)$, there exists some constants $\delta > 0$, $b > 0$ and $\varepsilon_0 \in (0,1)$ and $C = C(\mathtt{data}, a, \beta)$ such that if $\mathcal{A}_{p,\log}$ is (R_0,δ)-vanishing and Ω is (R_0,δ)-Reifenberg flat for some $R_0 > 0$, the following inequality $\omega\left(\mathcal{V}_{\varepsilon,\mathcal{λ}}\right) \leq C\varepsilon\omega\left(\mathcal{W}_{\mathcal{λ}}\right)$ holds for all $\mathcal{λ} > 0$ and $0 < \varepsilon < \varepsilon_0$, with $\mathcal{V}_{\varepsilon,\mathcal{λ}}, \mathcal{W}_{\mathcal{λ}}$ defined by $\mathcal{V}_{\varepsilon,\mathcal{λ}} := \{x \in \Omega : \mathbb{M}_\beta\mathbb{U}(x) > \varepsilon^{-a}\mathcal{λ}; \mathbb{M}_\beta\mathbb{F} \leq \varepsilon^b\mathcal{λ}\}$; $\mathcal{W}_{\mathcal{λ}} := \{x \in \Omega : \mathbb{M}_\beta\mathbb{U}(x) > \mathcal{λ}\}$.

Generalized weighted Lorentz estimate

Theorem (Generalized weighted Lorentz estimate)

Given $\beta \in [0, n)$, $w \in \mathbf{A}_\infty$ and $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{R}^+)$. With a function Γ defined as follows:

$$\Gamma(\tau) = \int_0^\tau \mu(s) ds, \quad \tau \in [0, \infty), \tag{10}$$

assume that there exists two constants $\varsigma_2 > \varsigma_1 > 1$ satisfying

$$\varsigma_1 \Gamma(\tau) \leq \Gamma(2\tau) \leq \varsigma_2 \Gamma(\tau), \tag{11}$$

for each $\tau \geq 0$. Then, one can find some constants $\delta = \delta(\mathtt{data}) > 0$, $C = C(\mathtt{data}, \beta, \varsigma_1, \varsigma_2) > 0$ such that if $\mathcal{A}_{p,\log}$ is (R_0, δ)-vanishing and Ω is (R_0, δ)-Reifenberg flat for some $R_0 > 0$, the following inequality

$$\left\| \mathbb{M}_\beta \mathbb{U} \right\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)} \leq C \left\| \mathbb{M}_\beta \mathbb{F} \right\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)}, \tag{12}$$

holds true for all $0 < \mathfrak{q} < \infty$ and $0 < \mathfrak{s} \leq \infty$ provided $\mathbf{M}_\beta \mathbb{F} \in \Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)$.

Proof of generalized weighted Lorentz estimate

Let's take a suitable $0 < a < \min\left\{\frac{1}{\mathfrak{q}}\log_2 \varsigma_1; 1\right\}$, apply above good-$\mathfrak{I}$ type inequality and distinguish two cases:

Case 1. $s < \infty$. The statement in (12) can be implied from the following estimate:

$$\left\|\mathbb{M}_\beta \mathbb{U}\right\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)} \leq C\left(\varsigma_2^{\log_2 C+2}\varsigma_1\right)^{\frac{1}{\mathfrak{q}}} \varepsilon^{\frac{1}{\mathfrak{q}}\log_2 \varsigma_1 - a} \left\|\mathbb{M}_\beta \mathbb{U}\right\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)} + C\varepsilon^{-a-b}\varsigma_2^{\frac{1}{\mathfrak{q}}} \left\|\mathbb{M}_\beta \mathbb{F}\right\|_{\Lambda^{\mathfrak{q},\mathfrak{s}}_{\omega,\mu}(\Omega)},$$

by choosing $\varepsilon = \varepsilon_0 > 0$ sufficiently small such that $C\left(\varsigma_2^{\log_2 C+2}\varsigma_1\right)^{\frac{1}{\mathfrak{q}}} \varepsilon_0^{\frac{1}{\mathfrak{q}}\log_2 \varsigma_1 - a} \leq \frac{1}{2}$.

Case 2. $s = \infty$. The statement in (12) can be implied from the following estimate:

$$\left\|\mathbb{M}_\beta \mathbb{U}\right\|_{\Lambda^{\mathfrak{q},\infty}_{\omega,\mu}(\Omega)} \leq C\left(\varsigma_2^{\log_2 C+2}\varsigma_1\right)^{\frac{1}{\mathfrak{q}}} \varepsilon^{\frac{1}{\mathfrak{q}}\log_2 \varsigma_1 - a} \left\|\mathbb{M}_\beta \mathbb{U}\right\|_{\Lambda^{\mathfrak{q},\infty}_{\omega,\mu}(\Omega)} + C\varepsilon^{-a-b}\varsigma_2^{\frac{1}{\mathfrak{q}}} \left\|\mathbb{M}_\beta \mathbb{F}\right\|_{\Lambda^{\mathfrak{q},\infty}_{\omega,\mu}(\Omega)},$$

by choosing $\varepsilon = \varepsilon_0 > 0$ sufficiently small such that $C\left(\varsigma_2^{\log_2 C+2}\varsigma_1\right)^{\frac{1}{\mathfrak{q}}} \varepsilon_0^{\frac{1}{\mathfrak{q}}\log_2 \varsigma_1 - a} \leq \frac{1}{2}$.

Our manuscript

Regularity results for a borderline case of elliptic double-phase operators

Minh-Phuong Tran[*,†], Thanh-Nhan Nguyen[‡], Tan-Phuc Nguyen[§]

June 10, 2025

Abstract

This article is devoted to studying the regularity properties of weak solutions to a class of quasilinear double-phase elliptic equations with logarithmic perturbation terms in divergence form, under suitable structural conditions on the given data. These problems pose significant challenges due to their non-uniform ellipticity and the presence of mixed polynomial–logarithmic growth behavior. Inspired by recent progress in non-standard growth theory, we derive decay estimates for the level sets of the gradient of weak solutions. These estimates, in turn, yield gradient regularity results in generalized weighted Lorentz spaces, provided that the coefficients satisfy some appropriate conditions. Our findings contribute to a deeper understanding of the behavior of solutions to variational problems with non-standard growth, with potential applications to the modeling of strongly anisotropic materials.

Keywords: Regularity estimates; Non-uniform ellipticity; Fractional maximal operators; Lorentz spaces

Future researches

The results and proof techniques developed in our work can be further improved and extended to address more complex situations, such as those arising when the function $\mathbb{H}_{q,\log}$ in (H) is replaced by the following function:

$$\mathbb{H}_{p,q,\alpha,\beta,\log}(x,\xi) := \mathrm{a}(x)|\xi|^p \log^{\alpha}(e+|\xi|) + \mathrm{a}(x)|\xi|^q \log^{\beta}(e+|\xi|), \quad (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Furthermore, the gradient estimates can be extended to more general function spaces, such as generalized Morrey spaces.

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Tan-Phuc Nguyen's Presentation

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Thank you for your attention!